

Measurement 3

Oscillatory Motion: The Mass-on-a-Spring System

Objects of the experiment:

- Measure the stiffness constant of a spring.
- Determine the frequency of vibration of a mass/spring system as a function of mass.
- Observe coupled oscillations.

Theory

In general, periodic or oscillatory motion is any motion in which the displacement (or an analogous quantity, such as angular displacement) varies periodically. The simplest example of an oscillatory motion is simple harmonic motion (SHM). In SHM the displacement varies according to a harmonic (sine or cosine) function and can be written as

$$x(t) = A \cos(\omega t - \phi) \quad (1)$$

Taking derivatives as a function of time gives the velocity and acceleration:

$$v(t) = dx/dt = -\omega A \sin(\omega t - \phi) = -v_{\max} \sin(\omega t - \phi) \quad (2)$$

$$a(t) \equiv d^2v/dt^2 = -\omega^2 A \cos(\omega t - \phi) = -a_{\max} \cos(\omega t - \phi) \quad (3)$$

Notice that

$$a(t) = \frac{d^2x(t)}{dt^2} = -\omega^2 x(t). \quad (4)$$

This is a very important equation. It states that in a SHM the acceleration is proportional to the displacement and opposite to it in direction. We can reverse the meaning of Eq. (4) and claim that whenever the dynamics of a system results in an accelerations that is proportional to the displacement and opposite to its direction, the resulting motion is SHM. Eq. (4) is the differential equation of SHM, that relates $x(t)$ to its second derivative. Eq. (1) is the most general solution of this differential equation. ω , the angular frequency, appears in Eq. (4); it is determined by the dynamics of the system. Consequently, a given system can only oscillate with one possible angular frequency.

The amplitude, A , and the phase, ϕ , are determined by how the oscillation is started. For example, if the vibrating object is pulled away from equilibrium by a distance A and released without initial velocity, the amplitude will be A and the phase ϕ will be zero.

The simplest system that exhibits SHM is a mass sliding on a frictionless level surface, attached to a horizontal spring. If the mass is moved out of the equilibrium point, the spring exerts a restoring force given by Hooke's law:

$$F = -kx \quad (5)$$

where k is the stiffness constant of the spring. Newton's second law results in the following differential equation for $x(t)$:

$$F_{\text{spring}} = -kx(t) = ma = m \frac{d^2x(t)}{dt^2}. \quad (6)$$

Comparison to Eq. (4) shows that this is the differential equation of a SHM and the angular frequency, ω , is given by

$$\omega = \sqrt{\frac{k}{m}}. \quad (7)$$

The period is given by

$$T = 2\pi / \omega = 2\pi \sqrt{\frac{m}{k}}. \quad (8)$$

Eqs. (5)-(8) are also valid for an object hanging from a spring and oscillating vertically, if its displacement is measured from the equilibrium point, where the force of gravity is balanced by a spring force due to the initial stretching of the spring.

In this experiment, you will determine the stiffness constant of a spring and then measure the period of oscillation for several masses attached to the spring. You will also measure directly the force applied by the spring and see if it has the expected magnitude and phase relationship to $x(t)$.

Procedure

Determination of the stiffness constant of the spring from static equilibrium

Equation (7) shows that the angular frequency of an oscillation depends on the mass of the oscillating object and the stiffness constant of the spring. The mass can be measured easily with a balance, while the stiffness constant can be determined by measuring the elongation of the spring under static load according to Hooke's law.

A force sensor will measure the load and a position sensors will measure the displacement. Both sensors are connected to your computer via the LabPro interface box: the force sensor to the CH1 port and the motion sensor to the DIG/SONIC1 port. When all connections have been made, power up the computer and select the "Oscillations" icon. A window should appear that shows three plots: Force, Position, and Velocity as a function of time.

This measurement involves *static* equilibrium, thus there should be no motion of the mass and spring. But slight vibrations are unavoidable, so you should take averages over several seconds, using the "STAT" button in the LoggerPro software's top menu bar.

Each bench has two springs, a "long" and a "short" one.

1. Attach the mass holder to the **long** spring and hang the assembly from the force sensor on the horizontal support beam. Leave the holder empty. Place the position sensor directly below the holder. The distance between the bottom of the holder and the position sensor must be at least 80 cm. In order to measure F as a function of x , it is necessary to establish a reference point at which the initial distance between the bottom of the holder and the position sensor is defined as zero and the force sensor is tared to zero. In this way, the sensor neglects the weight of the spring and the holder and measures only the additional weight placed on the holder. This is done by using the “Zero” button in the software’s top menu bar. Hit this button, then collect data for the preset time. Choose the “Erase and Continue” option that comes up the first time you attempt to collect data. You will see a plot of $F(t)$ and $x(t)$ that fluctuate slightly around zero.
2. Add 50 g and allow the holder to reposition itself until a new static equilibrium is established. Hit “Collect” to record new $F(t)$ and $x(t)$ plots. Highlight at least a 6 second range on each plot and use the “STAT” button to compute the mean and standard deviation of F and x for the applied mass.
3. Increase the load in 50 g steps up to 500 g and measure F and x for each load.
4. Make a linear plot of F versus x and perform least squares fitting to find the spring constant k and its uncertainty. (See whether all the points on the F versus x plot are on a straight line or some points – perhaps at low or high loads – deviate substantially from the straight line. If you find substantial deviation, exclude those points from the fit and determine k again using only the straight portion of the plot.)
5. Determine the random error of the force from the straight-line fit. You must have independent error estimates from when you collected the F and x data in steps 2 and 3. Compare these errors. Are they consistent with each other?

Measurements of simple harmonic motion

Procedure

You will measure $x(t)$ for each of five masses when they are individually attached to the spring and set into oscillatory motion by a small initial pull. You will use the Sonic Motion Detector to measure $x(t)$.

1. Measure the mass of the holder and the mass of the **long** spring. When the system oscillates, only the very end of the spring oscillates with the same amplitude as the attached mass, the amplitude of the other points is less. Detailed analysis shows that this can be taken into consideration by adding only **one third** of the mass of the spring to the oscillating mass.
2. Place the 200-g mass on the holder and wait until it comes to rest. Zero both sensors. Pull the mass down about 3 cm and release it to oscillate. Make sure that the oscillation is vertical; if there is a clear swinging component of the motion, repeat the run. A few seconds after moving away your hand, hit “Collect” and record an $x(t)$ curve for the preset time of 10 seconds. You will see a sinusoidal curve with 12 or more periods. If the curve is jagged or contains rapid jumps, hit “Collect” again until you get a fairly clean sinusoidal plot for $x(t)$. Determine the

period of your oscillation and its uncertainty. You can do this by measuring the time between maxima, minima, or zero crossings. Which one provides the most accurate result? Clearly describe your procedure. Calculate also the associated ω and its uncertainty.

3. The software is preprogrammed to plot $x(t)$ and $v(t)$, which your computer *calculates* from the $x(t)$ data by taking the slope to get instantaneous velocity. Determine the phase shift (in radians) between your $x(t)$ and $v(t)$ functions. Does this agree with theory? Measure the ratio of A to v_{max} . Does this agree with theory?

Measure the phase shift (in radians) between your $x(t)$ and $F(t)$ functions. Does this agree with theory? Measure the ratio of A to F_{max} . Does this agree with theory?

Make a printout of your entire screen before going to the next step.

4. Now add more mass in 50-g steps up to 400 g and repeat steps 2 through 3. Exceeding 400 g can saturate the sensor.

Make sure that at the end of these measurements, you have five printouts and five sets of data for angular frequency versus mass. Attach the printouts to your lab report. Make sure there is a time stamp on your printouts (this is the default setting).

Further calculations associated with SHM

1. One can linearize the relationship between m and ω (Eq. 7) as

$$\omega = \sqrt{k} \cdot \sqrt{1/m}$$

Therefore, plotting ω as a function of $\sqrt{1/m}$ should give a straight line. Perform least squares fitting to determine k . Compare this result to the value of k derived from the static measurements earlier. Remember to include the mass of the holder plus one third of the spring.

2. One can also linearize Eq. (7) by taking the logarithm of both sides:

$$\ln \omega = \frac{1}{2} \ln k - \frac{1}{2} \ln m$$

Make the corresponding plot of $\ln \omega$ versus $\ln m$ and perform least squares fitting to determine the power law behavior. How close does this come to square root behavior? Your fit also provides k and its error. Compare its result to the values of k found in the previous part and by the static experiment.

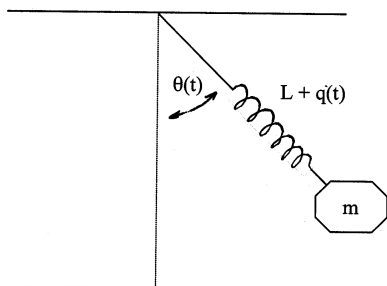
3. What are the largest sources of error for the determination of k from these data?

Demonstration of coupled oscillatory motion

Many oscillating systems are capable of oscillations in more than one way. Examples include bridges, molecules, and a variety of other systems, big and small. Your experimental setup is not

an exception. Although care was taken to avoid swinging in the previous parts of the experiment, swinging sideways like a pendulum is of course possible. As the forces and displacements of the two motions are not independent of each other – e.g. the period of the pendulum motion depends on the length of the spring that changes with the elastic oscillation – the two oscillations are coupled.

The exact mathematical treatment of this system looks formidable, but it will be a straightforward question in intermediate mechanics. If you restrict the motion to a plane (not necessarily the case, as swinging can take place in two mutually perpendicular directions) the position of the mass can be described by two quantities, $\theta(t)$ and $q(t)$, defined on the figure



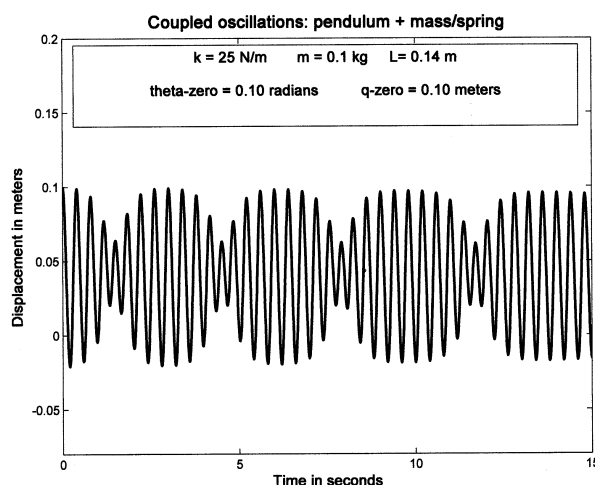
below:

The equations of motion can be derived in Lagrangian mechanics. They are:

$$\{L + q(t)\} \frac{d^2\theta(t)}{dt^2} + 2 \frac{dq(t)}{dt} \frac{d\theta(t)}{dt} + g \sin\theta(t) = 0$$

$$\frac{d^2q(t)}{dt^2} - \{L + q(t)\} \left(\frac{d\theta}{dt} \right)^2 + \frac{k}{m} q(t) - g \cos\theta(t) = 0.$$

These equations are too complicated for analytical solution (and will remain so at any level of mathematical sophistication) but they can be solved numerically without much difficulty. The result for $q(t)$ is a combination of two oscillations, a fast one and a slower one that serves as an “envelope.” A typical result is shown below.



Similar patterns appear frequently, if two oscillations with slightly different frequencies are combined. (In the current case, the two frequencies are associated with the pendulum motion and with the oscillation by stretching the spring.) The simplest combination of two oscillations is addition. Using a basic trigonometric identity

$$x(t) = \sin(\omega_1 t) + \sin(\omega_2 t) = 2 \sin\left(\frac{\omega_1 + \omega_2}{2} t\right) \cos\left(\frac{\omega_1 - \omega_2}{2} t\right)$$

Notice that this equation describes a “beating” signal quite similar to the one shown above. The fast oscillation is with the mean angular frequency and the amplitude modulation is described by the difference of the angular frequencies.

Experimental procedure

Although the coupled swinging and oscillation of a mass-on-a-spring system are difficult theoretically, you can easily demonstrate the basic phenomenon. As the above equation shows, choosing the right angular frequencies is important. If the two frequencies are too close, the amplitude modulation happens too slowly and the oscillation dampens before you could identify the pattern. If the frequencies are very different, you may be able to see ripples on a slow oscillation, but not the “beating” pattern obtained from the simulation. This is why you should use the **short** spring and 200 g added to the holder in this experiment.

Click on the “Experiment” Menu on the upper toolbar and choose “Extend collection” *twice*. This will allow you to record a run for sufficient time to be able to see the coupling behavior. “Zero” the sensors, then collect a dataset by pulling down and sideways on the spring a *small amount*, say 1 cm. If you do not see patterns similar to those shown below, add mass on the holder in increments of 10 g until you do. Once you have a good condition, re-run the experiment and look at the mass during the two portions of the motion when $x(t)$ shows maximum oscillation and when it shows minimum. What is the mass doing during each of these situations?

1. Make a printout of the waveforms and attach it to your lab report. Make sure there is a time stamp on your printout (this is the default setting).
2. Use your figure to determine the period and angular frequency of both the fast oscillations and the slow envelope. Calculate the angular frequencies of the component oscillations.